SCHRIFTELIJK (HER)TENTAMEN
COSMOLOGY
1^{st} term 2007/2008

NOTE: THIS EXAM CONTAINS 4 QUESTIONS and 6 pages.

Question 1.

A fundamental concept in cosmology, crucial for the analysis of cosmological observations, is *redshift*.

a) The metric of the spacetime obeying the cosmological principle and Weyl's principle is called the Robertson-Walker metric.
  - write out the metric form, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, for this spacetime
  - explain the meaning of each of the characteristic quantities. Focus in particular on those concerning the radius of curvature of spacetime, the related definition of the "scale factor" a_{exp} of the medium, and the factors expressing its curvature.

b) Derive the relation between redshift z and the expansion factor a of the Universe on the basis of the RW metric.

c) When you observe an object at redshift z the intrinsic timescale Δt of such an object has changed. Give an expression for this '*cosmological time dilation.* How would you be able to test the cosmological origin of such a time dilation ?

d) Imagine you observe a quasar at a cosmological redshift z_{cosm}. The quasar is moving a peculiar velocity v_{pec} with respect to the cosmological background, hence causing a Doppler redshift z_{Dopp}. In the center of the quasar there is a supermassive black hole. While climbing out of its deep potential well ϕ, radiation gets gravitationally redshifted to z_{grav}. Upon finally observing the radiation with a telescope on planet Earth you observe it with a total redshift z_{tot}. Give the expression for the total redshift z_{tot}.

e) While coordinate distance r is mainly a theoretical concept, redshift z is what is measured by a telescope. Show that one can translate a redshift z to the coordinate distance of an object by

$$r(z) = \frac{c}{H_0} \int_0^z \frac{dz'}{(H(z')/H_0)}$$

Given the expression for $r(z)$ one also can derive the expressions for $S_k(r)$. How are these essential relations for observational cosmology called ? (after the person who solved the integrals analytically).

f) Often we wish to know the number of objects within a redshift interval $[z, z + dz]$. If the (comoving) space density N_0 of a particular class of object does not change, derive from the RW metric how many objects within a redshift internval dz you will find. (Hint: express in terms of present-day density N_0, the radius of curvature R_c and curvature term $S_k(r/R_c)$).

Question 2.

Key cosmological factors are the cosmological density parameter $\Omega(t)$, the Hubble parameter $H(t)$, the cosmological constant Λ abd the curvature parameter k.

a) Give the expressions for the Friedman-Robertson-Walker-Lemaitre equations and indicate the meaning for each factor in the FRW equations.

b) For a FRW Universe:
   - define the meaning of the critical density ρ_{crit}.
   - give the expression for the critical density.
   - give an estimate of the value of the present-day critical density of the universe, in $[g/cm^3]$ and in $[M_\odot/Mpc^3]$.

c) Subsequently, provide the definition of the cosmological density parameter $\Omega(t)$.

d) Provide the definition for the Hubble parameter $H(t)$. What are the usual units in which we express $H(t)$? What are the present-day estimates for H_0 ?

e) Give the definition of acceleration parameter q. Derive its value for a pure matter-dominated Universe, in terms of Ω_m. Same for a Universe with matter and a cosmological constant, in terms of Ω_m and Ω_Λ.

f) Derive the expression of the curvature parameter k in terms of Ω and H.

Question 3.

In this question we will investigate some major transitions in the history of the Universe. Some will be of dynmical nature, some of a thermodynamic nature.

To reconstruct the thermal history of the Early Universe, one needs to combine the knowledge of the temperature evolution of the Universe, that of the interaction rate Γ of the various relevant physical processes and the dynamical timescale of the Universe (...). Physical transitions occur when physical processes get out of equilibrium.

Let us first consider the cosmic energy equation,

$$\dot{\rho} + 3\,(\rho + \frac{p}{c^2})\,\frac{\dot{a}}{a} = 0$$

his follows directly and implicitly from the FRW equation. It describes the time evolution of the energy density ρ of the various components of the Universe: radiation, matter and dark energy.

a) Argue why the energy equation implies the expansion of the Universe to be adiabatic.

b) Given that the expansion of the Universe is adiabatic, show that temperature of radiation evolves inversely to the expansion factor of the Universe,

$$T(t) \propto \frac{1}{a(t)}$$

Note: the adiabatic index for radiation is $\gamma = 4/3$.

c) What is the dynamical timescale of the Universe ? Write down the criterion for a physical process being in equilibrium and thus for when it runs out of equilibrium.

d) What makes the Universe such a very special physical system ? In this, take into account that the temperature is continuously decreasing (affecting the reaction rates and timescales !) and about the number of photons available (a fundamental number of cosmology !!!!).

e) Use the energy equation to derive the time evolution $\rho(a)$ of radiation, matter ("dust" with pressure $p = 0$) and dark energy as a function of expansion factor $a(t)$. For dark energy distinguish two cases: pure cosmological constant $p = -\rho c^2$, and more general dark energy $p = w\rho c^2$, $-1 \leq w < -1/3$.

f) A major transition of the early Universe is that of the "radiation-matter" transition. Explain what this is, compute at which redshift this happens (as a function of Ω_r, Ω_m and H_0). For the concordance cosmological model ($\Omega_r = 8.4 \times 10^{-5}$, $\Omega_m = 0.27$ and H_0=71 km/s/Mpc).

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g) Another important dynamical transition of the Universe is that between a matter-dominated Universe and a Universe dominated by the cosmological constant. One may distinguish two definitions: the stage at which the Universe turns from decelaration to acceleration, the other one the stage at which the dark energy density in the Universe becomes larger than that of matter. Derive expressions for the redshift of both (for the acceleration one: look at the acceleration part of the FRW equation, see question 2). Subsequently, give the value for these transition times for concordance cosmology: $\Omega_m = 0.27$ and $\Omega_\Lambda = 0.73$.

h) At around a temperature of $T \approx 3000K$ the Universe, and in particular the blackbody cosmic radiation, undergoes a major transition. This, perhaps most important cosmological transition, includes three closely related processes. Describe each of these, in some detail, and describe qualitatively what happened. You may use some drawings and sketches.

i) According to a simple equilibrium evaluation on the basis of the Saha equation the transition should have happened at a temperature of $T_\gamma = 3740K$. Why did it take place much later, at $T_\gamma \approx 3000K$?

Question 4.

On the basis of the FRW equations (see question 2), we are going to explore the time evolution of the Hubble parameter and the cosmic density parameter.

a) On the basis of the FRW equations - taking into account the time evolution $\rho(a)$ of the constituents of universe (radiation, matter, cosmological constant, curvature) - derive the expression for the Hubble parameter $H(a)$ (or, rather, for $H(a)^2$). Note: write the expression for $H(a)$ in terms of a, H_0, $\Omega_{m,o}$, $\Omega_{r,0}$ and $\Omega_{\Lambda,0}$ (and express curvature k in terms of $\Omega_0 = \Omega_{m,0} + \Omega_{r,0} + \Omega_{\Lambda,0}$: see question 2f).

b) For the specfic case of a pure matter-dominated Universe ($\Omega_{r,0} = \Omega_{\Lambda,0} = 0$) derive the specific expression for the Hubble parameter in terms of redshift z. Finally, give the expression for an Einstein-de Sitter Universe.

c) Given the expression for the Hubble parameter $H(a)$, with $\dot{a} = H(a)a$, we can dervie the general expression for cosmic time $t(z)$ at any expansion factor a or redshift z. Derive this integral expression (again, for a generic Universe with $\Omega_{r,0}$, $\Omega_{m,0}$, $\Omega_{\Lambda,0}$).

d) Give the expression for cosmic time $t(z)$ for the explicit case of a matter-dominated Universe. Finally, calculate explicitly the cosmic time $t(z)$ for an Einstein-de Sitter Universe.

e) For a matter-dominated Universe with $\Omega_{m,0} = \Omega_0$ derive from the FRW equations the time evolution of the cosmic density factor $\Omega(z)$. That is, needed is to derive an expression for $\Omega(z)$ in terms of $\Omega_0 = \Omega_{m,0}$ and redshift z.

f) For a matter-dominated Universe show that if $\Omega_0 = 1$ then $\Omega(t) = 1$ always. That is, an Einstein-de Sitter Universe always remains an Einstein-de Sitter Universe.

g) Show that for $a \downarrow 0$ that $\Omega \to 1$, for $\Omega_0 > 1$, $\Omega_0 < 1$ and $\Omega_0 = 1$. Make a sketch of the generic evolution of $\Omega(a)$ as a function of expansion factor a. Given the resulting evolution of Ω, explain the flatness problem of a matter-dominated Universe.

h) For a one-component Universe (with curvature $k \neq 0$, ie. $k = 1$ or $k = 1$), show that

$$|\Omega(t) - 1| = \frac{c^2}{R_0} \dot{a}^{-2}$$

i) Use the energy equation (question 3) and the FRW equation to show that the generic time evolution $a(t)$ for a one-component Universe with equation of state $\rho = w\rho c^2$ ($w > -1$) is

$$a(t) \propto t^{2/(3+3w)}$$

j) Inserting the time evolution of (i) into the expression for $\Omega(t)$ in (h), derive the expression for the time evolution of $|\Omega(t) - 1|$. Argue why for $w < -1/3$ the trend of the evolving Ω is opposite to that of $w > -1/3$, i.e $|\Omega(t) - 1|$ decreases as the Universe expands. Explain why this solves the flatness problem (at inflation the Universe has the same exponential de Sitter expansion as for a Universe with cosmological constant).

SUCCES !!!!
BEDANKT VOOR JULLIE AANDACHT EN INTERESSE DIT KWARTAIR !!!!

Rien